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EXPLICIT ASYMPTOTIC SOLUTION FOR THE MAXIMUM-ENERGY-RELEASE-RATE PROBLEM†

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Abstract—An explicit 3-term asymptotic solution is obtained for the problem for small branch angles. Comparison with the numerical solution shows that the asymptic formulae are reasonably accurate for branch angles as large as 72°.

1. INTRODUCTION

The determination of stress-intensity factors for a crack with an infinitesimal branch at a tip is important in studying the branching phenomenon, and has been extensively studied by many researchers (e.g. [1-7]). A discussion of the various approaches may be found in [7]. The result of the asymptotic analysis carried out in [6] is a single integral equation which enables the direct evaluation of the stress-intensity factors at the tip of the infinitesimal branch without assigning a small length for the branch. This integral equation was solved numerically in [7].

The purpose of this paper is to show that the aforementioned integral equation can be solved by a perturbation procedure, using the branch angle as a small parameter. The result is an explicit 3-term approximate solution which agree reasonably well with the numerical solution for branch angles up to 72°.

2. STATEMENT OF THE PROBLEM

Consider a Z-shaped crack characterized by a main crack length $2a$, a branch crack length ϵa ($0 < \epsilon \ll 1$), and a branch crack angle $\alpha\pi$. We shall use "a" as a length scale so that the crack configuration in the dimensionless (x_1, x_2)-plane assumes the form depicted in Fig. 1. The infinite plane is characterized by a shear modulus μ and Poisson's ratio ν, and is subjected to a uniform load applied at infinity. Using $a^2\mu$ as a force scale, the condition at infinity is

$$\tau_{\alpha\beta} = \sigma_{\alpha\beta} \quad \text{as} \quad |x_1{}^2 + x_2{}^2| \to \infty \tag{2.1}$$

where $\sigma_{\alpha\beta}$ are the values of the dimensionless stresses $\tau_{\alpha\beta}$ at infinity.

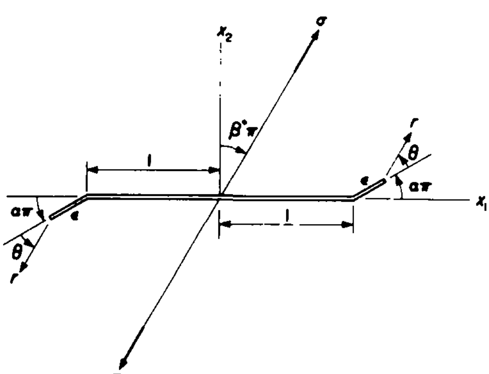

Fig. 1.

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Let the solution near the tips of the Z be described by

$$\tau_{11}+\tau_{22}=\frac{2}{\sqrt{(2\pi r)}}\left\{k_1(\sigma,\alpha,\epsilon)\cos\frac{\theta}{2}-k_2(\sigma,\alpha,\epsilon)\sin\frac{\theta}{2}\right\} \tag{2.2}$$

where (r,θ) are the near-tip polar coordinates indicated in Fig. 1. Then $k_{1,2}(\sigma,\alpha,\epsilon)$ are the stress-intensity factors at the tips of the Z. The limits

$$K_{1,2}(\sigma,\alpha)=k_{1,2}(\sigma,\alpha,0) \tag{2.3}$$

are the asymptotic approximations of the true stress-intensity factors $k_{1,2}$; and the limits

$$K_{I,II}(\sigma)=\lim_{\epsilon\to 0}k_{1,2}(\sigma,0,\epsilon) \tag{2.4}$$

are just the exact stress-intensity factors associated with the linear crack of length 2.

It was shown in [6] that

$$K_1(\sigma,\alpha)+iK_2(\sigma,\alpha)=\sqrt{(\pi)}\,e^{i\alpha\pi}\left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha/2}(\sigma_{22}+i\sigma_{12})\overline{\psi'(-\alpha)} \tag{2.5}$$

where the complex function $\psi(z)$ is the solution of the integral equation

$$\psi(z)=\frac{1}{4\pi i}(e^{i2\alpha\pi}-1)\frac{\sigma_{22}+i\sigma_{12}}{\sigma_{22}-i\sigma_{12}}\int_C\frac{(\zeta^2-1)\overline{\psi'(\bar{\zeta})}}{(\zeta+\alpha)(\zeta-z)}\,d\zeta+z \tag{2.6†}$$

and the arc C is defined by

$$C:\quad z=e^{i\theta}\qquad \pi\le\theta\le 2\pi. \tag{2.7}$$

For the case of a uni-directional load depicted in Fig. 1 and defined by

$$\sigma_{22}=\sigma\cos^2\beta^*\pi,\qquad \sigma_{12}=\sigma\cos\beta^*\pi\sin\beta^*\pi \tag{2.8‡}$$

(2.6) may be put into the more convenient form

$$\psi(z)=\frac{\sin\alpha\pi}{2\pi}e^{i(\alpha+2\beta^*)\pi}\int_C\frac{(\zeta^2-1)\overline{\psi'(\bar{\zeta})}}{(\zeta+\alpha)(\zeta-z)}\,d\zeta+z. \tag{2.9}$$

Equation (2.9) has been solved numerically and the relevant results may be found in [7]. The purpose of this paper is to show that for small values of α eqn (2.9) may be solved by a perturbation procedure. Specifically, a 3-term explicit formula is obtained for the quantity $\psi'(-\alpha)$ needed in (2.5). It is

$$\psi'(-\alpha)=1-\frac{\sin\alpha\pi}{2\pi}e^{i(\alpha+2\beta^*)\pi}\left(\frac{2\alpha}{1-\alpha^2}+\ln\frac{1+\alpha}{1-\alpha}+i\pi\right)+\left(\frac{\sin\alpha\pi}{2\pi}\right)^2\left(\frac{\pi^2}{2}-4\right)+\cdots. \tag{2.10}$$

We shall see in the following section that the form of (2.10) comes out naturally from the development although it is accurate only to the order of α^2. Substituting (2.10) into (2.5) we obtain:

†The complex z-plane is not the (x_1, x_2)-plane indicated in Fig. 1.

‡The parameter $\beta = 1/2-\beta^*$ was used in [7]. For the case of a crack-parallel shear τ, set $\beta^* = 1/2$ and $\sigma\cos\beta^*\pi=\tau$ in all equations.

Crack perpendicular σ_{22}

$$\frac{1}{\sigma_{22}\sqrt{\pi}}[K_1+iK_2]=i\left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha/2}e^{i\alpha\pi}\left[1-\frac{\sin\alpha\pi}{2\pi}e^{-i\pi\alpha}\left(\frac{2\alpha}{1-\alpha^2}+\ln\frac{1+\alpha}{1-\alpha}-i\pi\right)+\left(\frac{\sin\alpha\pi}{2\pi}\right)^2\left(\frac{\pi^2}{2}-4\right)\right]. \tag{2.11}$$

Crack-parallel σ_{12}

$$\frac{1}{\sigma_{12}\sqrt{\pi}}[K_1+iK_2]=\left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha/2}e^{i\alpha\pi}\left[1+\frac{\sin\alpha\pi}{2\pi}e^{-i\pi\alpha}\left(\frac{2\alpha}{1-\alpha^2}+\ln\frac{1+\alpha}{1-\alpha}-i\pi\right)+\left(\frac{\sin\alpha\pi}{2\pi}\right)^2\left(\frac{\pi^2}{2}-4\right)\right]. \tag{2.12}$$

Stress-intensity factors computed from these two formulae are tabulated and compared with the corresponding numerical results[7] in Table 1. The comparisons exhibit excellent agreement for $\alpha<0.2$ (36° branch angle), and reasonable agreement can be traced to values of α as large as 0.4 (72° branch angle).

Let $G(\sigma,\alpha)$ be the line-to-Z energy release rate. It may be shown by using Irwin's crack-tip-opening-displacement calculation[8] that

$$G(\sigma,\alpha)=\frac{1}{8}(\kappa+1)[K_1^2(\sigma,\alpha)+K_2^2(\sigma,\alpha)]. \tag{2.13}$$

where

$$\kappa=\begin{cases}3-4\nu & \text{plane stress}\\ (3-\nu)/(1+\nu) & \text{plane strain.}\end{cases} \tag{2.14}$$

The energy release rate can also be calculated in terms of the properties of the solution at infinity. This was done in [6] and the result is

$$G(\sigma,\alpha)=\frac{\pi}{8}(\kappa+1)(\sigma_{22}^2+\sigma_{12}^2)\left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha}\left(1+\frac{Q+\bar{Q}}{1-\alpha^2}\right) \tag{2.15}$$

Table 1.

α	*Crack-Perpendicular σ_{22}				**Crack-Parallel σ_{12}			
	Numerical Solution		Eq. (2.11)		Numerical Solution		Eq. (2.12)	
	K_1	K_2	K_1	K_2	K_1	K_2	K_1	K_2
0.00	1.0000	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	1.0000
0.02	0.9985	0.0314	0.9985	0.0314	-0.0941	0.9969	-0.0942	0.9969
0.04	0.9940	0.0625	0.9941	0.0626	-0.1877	0.9877	-0.1877	0.9877
0.06	0.9867	0.0932	0.9867	0.0935	-0.2800	0.9724	-0.2802	0.9724
0.08	0.9766	0.1232	0.9764	0.1239	-0.3707	0.9513	-0.3710	0.9512
0.10	0.9636	0.1523	0.9633	0.1537	-0.4590	0.9245	-0.4596	0.9241
0.12	0.9480	0.1803	0.9473	0.1826	-0.5444	0.8921	-0.5454	0.8914
0.14	0.9298	0.2070	0.9287	0.2105	-0.6265	0.8546	-0.6280	0.8533
0.16	0.9092	0.2323	0.9073	0.2373	-0.7047	0.8123	-0.7068	0.8100
0.18	0.8864	0.2559	0.8834	0.2628	-0.7787	0.7655	-0.7814	0.7619
0.20	0.8614	0.2777	0.8571	0.2868	-0.8479	0.7147	-0.8513	0.7094
0.22	0.8346	0.2976	0.8284	0.3092	-0.9121	0.6602	-0.9161	0.6527
0.24	0.8061	0.3155	0.7976	0.3299	-0.9710	0.6026	-0.9754	0.5924
0.26	0.7760	0.3313	0.7648	0.3487	-1.0240	0.5424	-1.0289	0.5288
0.28	0.7447	0.3450	0.7302	0.3645	-1.0716	0.4801	-1.0763	0.4625
0.30	0.7123	0.3565	0.6940	0.3801	-1.1131	0.4161	-1.1173	0.3940
0.32	0.6790	0.3658	0.6563	0.3925	-1.1148	0.3510	-1.1518	0.3237
0.34	0.6451	0.3729	0.6175	0.4026	-1.1177	0.2853	-1.1795	0.2522
0.36	0.6106	0.3778	0.5778	0.4103	-1.2003	0.2195	-1.2003	0.1802
0.38	0.5760	0.3806	0.5373	0.4157	-1.2170	0.1542	-1.2143	0.1081
0.40	0.5412	0.3813	0.4964	0.4186	-1.2276	0.0897	-1.2214	0.0365

*Multiply $\sigma_{22}\sqrt{\pi}$ to obtain physical values. $K_1(-\alpha)=K_1(\alpha)$ and $K_2(-\alpha)=-K_2(\alpha)$.

**Multiply $\sigma_{12}\sqrt{\pi}$ to obtain physical values. $K_1(-\alpha)=K_1(\alpha)$ and $K_2(-\alpha)=K_2(\alpha)$.

where

$$Q = \lim_{z\to\infty} z[\psi(z) - z] = -\frac{\sin \alpha\pi}{2\pi} e^{i(\alpha+2\beta^*)\pi} \int_C \frac{(\zeta^2-1)\overline{\psi'(\bar{\zeta})}}{\zeta+\alpha} d\zeta. \tag{2.16}$$

Substituting (2.5) and (2.14) into (2.13), we find

$$\psi'(-\alpha)\overline{\psi'(-\alpha)} = 1 + \frac{1}{1-\alpha^2}(Q+\bar{Q}). \tag{2.17}$$

The quantity Q consistent with the accuracy of (2.10) is

$$Q = \frac{\sin \alpha\pi}{2\pi} e^{i(\alpha+2\beta^*)\pi}(1-\alpha^2)\left(\frac{2\alpha}{1-\alpha^2} + \ln\frac{1+\alpha}{1-\alpha} + i\pi\right) + \left(\frac{\sin \alpha\pi}{2\pi}\right)^2(\pi^2-4) + \cdots. \tag{2.18}$$

Using (2.10) and (2.18), we find that (2.17) is satisfied to the order of α^2.

3. ASYMPTOTIC SOLUTION

Consider the complex function $\psi(z)$ satisfying the integral equation (2.9). The function is holomorphic in the whole z-plane cut along the semi-circle C defined by (2.7). The points $z = \pm 1$ are the images of the physical locations where an ϵ-branch meets the main crack. We have the following correspondence:

$$z = \begin{cases} -1 & \begin{cases}\text{a re-entrant corner} \\ \text{a regular corner}\end{cases} \quad \text{if } \alpha \lessgtr 0 \\ -\alpha & \text{a tip of the } Z \\ +1 & \begin{cases}\text{a regular corner} \\ \text{a re-entrant corner}\end{cases} \quad \text{if } \alpha \lessgtr 0. \end{cases} \tag{3.1}$$

In the neighborhood of $z = \pm 1$, the integral in (2.9) is coupled with the l.h.s., no matter how small the parameter α is. This coupling yields the corner behavior discussed in [7]. It may be interpreted as a kind of boundary-layer expansion in terms of the small parameter α. For $|z \pm 1| > 0$, the contribution of the integral in (2.9) is small if α is assumed to be small. This suggests that a regular perturbation procedure may be introduced to solve (2.9).

For convenience we introduce a small parameter $\delta = \sin \alpha\pi/2\pi$, and seek a solution of (2.9) in the form

$$\psi(z) = \sum_{n=0}^{\infty} \delta^n \psi_n(z, \alpha). \tag{3.2}$$

Substituting (3.2) into (2.9), we find that $\psi(z)$ may be written as

$$\psi(z) = \sum_{\text{even}} \delta^n \phi_n(z, \alpha) + e^{i(\alpha+2\beta^*)\pi} \sum_{\text{odd}} \delta^n \phi_n(z, \alpha) \tag{3.3}$$

where

$$\phi_0(z, \alpha) = z \tag{3.4}$$

and

$$\phi_{n+1}(z, \alpha) = \int_C \frac{(\zeta^2-1)\overline{\phi_n'(\bar{\zeta}, \alpha)}}{(\zeta+\alpha)(\zeta-z)} d\zeta. \tag{3.5}$$

Thus, the functions ϕ_n can all in principle be determined.

Integrating (3.5) by parts, we get

$$\phi_{n+1}(z,\alpha)=\frac{1}{z+\alpha}[f_{n+1}(z,\alpha)-f_{n+1}(-\alpha,\alpha)] \tag{3.6}$$

where

$$f_{n+1}(z,\alpha)=(z^2-1)\int_C \frac{\overline{\phi_n(\bar{\zeta},\alpha)}}{(\zeta-z)^2}\,\mathrm{d}\zeta. \tag{3.7}$$

The two quantities $\psi'(-\alpha)$ and Q are just

$$\psi'(-\alpha)=1+\sum_{2,4,6}\delta^n\phi_n'(-\alpha,\alpha)+\mathrm{e}^{i(\alpha+2\beta^*)\pi}\sum_{1,3,5}\delta^n\phi_n'(-\alpha,\alpha) \tag{3.8}$$

$$Q=\sum_{2,4,6}\delta^n Q_n(\alpha)+\mathrm{e}^{i(\alpha+2\beta^*)\pi}\sum_{1,3,5}\delta^n Q_n(\alpha) \tag{3.9}$$

where

$$\phi_n'(-\alpha,\alpha)=\frac{1}{2}f_n''(-\alpha,\alpha) \tag{3.10}$$

$$Q_n(\alpha)=\int_C \overline{\phi_{n-1}(\bar{\zeta},\alpha)}\,\mathrm{d}\zeta-f_n(-\alpha,\alpha). \tag{3.11}$$

The function $f_1(z,\alpha)$ can be integrated straightforwardly. It is

$$f_1(z,\alpha)=(z^2-1)\ln\frac{z-1}{z+1}+2z \tag{3.12}$$

where the logarithmic term is defined in accordance with the cut along C. Equations (3.10)–(3.12) yield

$$\phi_1'(-\alpha,\alpha)=\frac{2\alpha}{1-\alpha^2}+\ln\left|\frac{1+\alpha}{1-\alpha}\right|+i\pi \tag{3.13}$$

$$Q_1(\alpha)=(1-\alpha^2)\left(\frac{2\alpha}{1-\alpha^2}+\ln\left|\frac{1+\alpha}{1-\alpha}\right|+i\pi\right). \tag{3.14}$$

Unfortunately, the appearance of the logarithmic term in (3.12) makes the explicit evaluation of the higher order terms impossible. However, if the purpose is to determine $\psi'(-\alpha)$ and Q to the order of α^2, only the values of $\phi_2'(-\alpha,\alpha)|_{\alpha=0}$ and $Q_2(\alpha)|_{\alpha=0}$ are needed. These two quantities can be explicitly evaluated. We have

$$\phi_2'(0,0)=\frac{1}{2}f_2''(0,0)=\int_C\frac{\overline{\phi_1(\bar{z},0)}}{z^2}\,\mathrm{d}z-3\int_C\frac{\overline{\phi_1(\bar{z},0)}}{z^4}\,\mathrm{d}z \tag{3.15}$$

$$Q_2(0)=\int_C\overline{\phi_1(\bar{z},0)}\,\mathrm{d}z+\int_C\frac{\overline{\phi_1(\bar{z},0)}}{z^2}\,\mathrm{d}z. \tag{3.16}$$

For z on C, (3.6) and (3.12) imply that

$$\begin{aligned}\overline{\phi_1(\bar{z},0)}&=\frac{z^2-1}{z}\,\overline{\ln\frac{\bar{z}-1}{\bar{z}+1}}+2-\frac{i\pi}{z}\\&=\frac{z^2-1}{z}\left(\ln\frac{z-1}{z+1}-2\pi i\right)+2-\frac{i\pi}{z}.\end{aligned} \tag{3.17}$$

Substituting (3.17) into (3.15) and (3.16), and integrating we obtain

$$\phi_2'(0,0) = \frac{\pi^2}{2} - 4, \qquad Q_2(0) = \pi^2 - 4. \tag{3.18}$$

This concludes the derivation of (2.10) and (2.23).

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